We will prove the following 3 assertions by simultaneous induction:

I. $\pi_n S^n \cong \mathbb{Z}$ $\forall n \geq 2$ (Note that this is already known for $n=1$.)

II. HOMOTOPY ADDITION THEOREM:

$x_0 \in A \subseteq X$, A and X are path-connected

$g: (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$

$f_i: D^n \longrightarrow D^n$ $i = 1, \dots, k$, $f_i(\mathring{D}^n)$ disjoint open sets, $f_i|_{\mathring{D}^n}: \mathring{D}^n \xrightarrow{\cong} f_i(\mathring{D}^n)$

$g\left(\mathring{D}^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{D}^n)\right) \subseteq A$

$\gamma_i: I \longrightarrow A$, $\gamma_i(0) = g \circ f_i(*)$, $\gamma_i(1) = x_0$

$\Rightarrow \sum_{i=1}^{k} \mathrm{ldeg}(f_i)\,(\gamma_i)_* [g \circ f_i] = [g]$ in $\tilde{\pi}(X, A, x_0)$

III. HUREWICZ THEOREM (RELATIVE):

$n \geq 2$, X, A path-connected, $x_0 \in A \subseteq X$

$\pi_1(A, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$ (hence $\pi_1(X, A, x_0) = 0$)

$\pi_i(X, A, x_0) = 0$ $\forall i \leq n-1$

$\Rightarrow$ $H_i(X, A) = 0$ $\forall i \leq n-1$

& $\tilde{\pi}_n(X, A, x_0) \xrightarrow[\cong]{h} H_n(X, A)$

We have the ABSOLUTE HUREWICZ THM as an immediate consequence of III:

$\pi_1(X) = 0$, $n \geq 2$, $\pi_i(X, x_0) = 0$ $\forall i \leq n-1 \Rightarrow \tilde{H}_i(X) = 0$ $\forall i \leq n-1$ & $\pi_n(X, x_0) \cong H_n(X)$

(take $A = *$ and note that $\forall i > 0$: $H_i(X, *) \cong H_i X$)

We show that

$$\pi_{n-1} S^{n-1} \cong \mathbb{Z} \overset{A}{\Longrightarrow} \text{HAT in dim } n \overset{B}{\Longrightarrow} \text{Hurewicz in dim } n \overset{C}{\Longrightarrow} \pi_n S^n \cong \mathbb{Z}$$

C Apply the absolute Hurewicz thm to $X = S^n$:

$\pi_i S^n = 0$ $\forall i \leq n-1$ $\Rightarrow$ $\pi_n S^n \cong H_n S^n \cong \mathbb{Z}$.

Ⓐ 1

Lemma 0. $\pi_{n-1} S^{n-1} \cong \mathbb{Z}$, $f: (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$

Then $\deg f = 1$ iff $f \simeq id$, $\deg f = -1$ iff $f \simeq$ reflection.

PROOF [Waldhausen p. 7–9]: The directions "$\Leftarrow$" are trivial.

$n=1$: $f(\partial D^1) = \partial D^1$. $\lightning$ otherwise $f(\partial D^1) = pt$, and thus $f \underset{rel\ \partial D^1}{\simeq}$ const since $\pi_1 D^1 = 0$. $\Rightarrow \deg f = 0$ $\lightning$

The assertion follows by distinguishing b/w the cases

- $f(1) = 1$, $f(-1) = -1$
- $f(-1) = 1$, $f(+1) = -1$

$n \geq 2$: $D^n \simeq *$ $\Rightarrow$ f is a htp eq of pairs iff $f|_{\partial D^n}: \partial D^n \longrightarrow \partial D^n$ is a htp eq.

Hence $f \sim id$ iff $f|_{\partial D^n} \sim id_{\partial D^n}$,

$f \sim refl$ iff $f|_{\partial D^n} \sim refl_{\partial D^n}$.

$$\begin{array}{ccccccc}
0 = H_n D^n & \longrightarrow & H_n(D^n, \partial D^n) & \xrightarrow[\partial]{\cong} & H_{n-1}(\partial D^n) & \longrightarrow & H_{n-1} D^{n-1} = 0 \\
 & & \cong \downarrow f_* & & \cong \downarrow (f|_{\partial D^n})_* & & \\
0 = H_n D^n & \longrightarrow & H_n(D^n, \partial D^n) & \xrightarrow[\partial]{\cong} & H_{n-1}(\partial D^n) & \longrightarrow & H_{n-1} D^{n-1} = 0
\end{array}$$

$\deg f = \pm 1$ $\Rightarrow$ f_* is an iso $\Rightarrow$ $(f|_{\partial D^n})_*$ is an iso

$\pi_{n-1} S^{n-1} \xrightarrow{h} H_{n-1} S^{n-1}$ is surjective because we can get a generator of $H_{n-1} S^{n-1} \cong \mathbb{Z}$ in the image of h: for $[id] \in \pi_{n-1} S^{n-1}$, $h([id]) = [id_*(\iota)] = \iota$

By assumption h is a $\mathbb{Z} \twoheadrightarrow \mathbb{Z}$ map $\Rightarrow$ h is an iso.

h does the same as deg for this f, thus the assertion follows.

Lemma 1. $(\gamma_i)_*[g\circ f_i] = (\gamma_i')_*[g\circ f_i]$.

Consequently, the sum in HAT is independent of the paths γ_i.

Lemma 2. $g \sim g'$ as maps of pairs $\Rightarrow [g]=[g']$ in $\tilde{\pi}(X,A,x_0)$

Lemma 3. $f_i \underset{H}{\simeq} f_i'$, $H(\partial D^n \times I) \subseteq g^{-1}A$, γ_i' the path corresponding to f_i'

$\Rightarrow (\gamma_i)_*[g\circ f_i] = (\gamma_i')_*[g\circ f_i']$

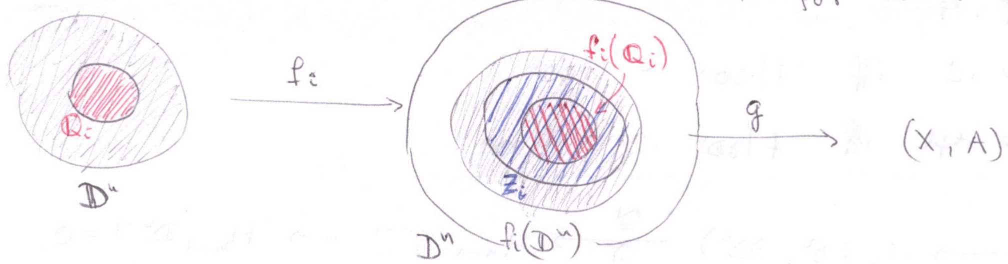
All three lemmata are direct consequences of the def of $\tilde{\pi}(X,A,x_0)$.

Prop. $Z_i \subseteq f_i(\mathbb{D}^n)$ closed balls (in the target-D^n)

$\Rightarrow \exists g' : (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$, $g\left(D^n \setminus \bigcup_{i=1}^{k} \mathring{Z}_i\right) \subseteq A$,

$\exists H : D^n \times I \longrightarrow X$, $g \underset{H}{\simeq} g'$, $H|_{\left(D^n \setminus \bigcup_{i=1}^{k} f_i(\mathbb{D}^n)\right)\times I} = g \circ pr_1$

PROOF: [Waldhausen p. 13]: Suffices to do it for $k=1$. (then induction)

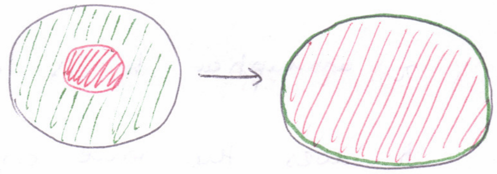


$0 \in Q_i \subseteq \mathbb{D}^n$, $f_i(Q_i) \subseteq \mathring{Z}_i$

$$D^n = \mathbb{D}^n \bigcup_{\partial \mathbb{D}^n} D^n \setminus f_i(\mathbb{D}^n)$$

Let H be the homotopy that:

- does nothing on $D^n \setminus f_i(\mathbb{D}^n)$
- on $\mathbb{D}^n$: the htp from $g\circ f_i$ given by radial projection on $\mathbb{D}^n \setminus Q_i$, radial homeo on Q_i



$g' := H(-,1)$

□

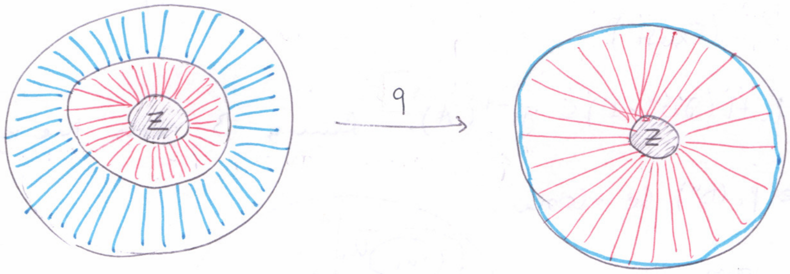
Ⓐ 3

Case $k = 1$.

Prop. $\Rightarrow$ replace g by g'. We have $[g] = [g']$ and by Lemma 2:

$\gamma_{1*}[g \circ f_1] = \gamma_{1*}[g' \circ f_1]$

Let $Z \subseteq \mathring{Z}_1$ be a smaller ball (around 0).



$q|_Z = id_Z$

$Q : D^n \times I \longrightarrow D^n$ homotopy between q and id_{D^n}

$f_1' := q \circ f_1$, Q gives a htp b/w f_1 and f_1' (need not be constant on ∂D^n) as a htp of maps $(D^n, \partial D^n) \longrightarrow (D^n, D^n \setminus \mathring{Z}_1)$

$g'(D^n \setminus \mathring{Z}_1) \subseteq A \Rightarrow$ We have a homotopy b/w $g' \circ f_1$ and $g' \circ f_1'$ as htp of maps $(D^n, \partial D^n) \longrightarrow (X, A)$

$\Rightarrow g' \circ f_1$ can be replaced by $g' \circ f_1'$ (Lemma 3.)

Sublemma. $\mathrm{ldeg}\, f_1 = \deg f_1'$

Pf.

$$\begin{tikzcd}
\mathbb{Z}_n\, H_n(D^n, \partial D^n) \arrow[r, "\cong"] \arrow[d, "f_{1*}"] & H_n(D^n, D^n \setminus x) \arrow[d, "f_{1*}", "\cong"'] \arrow[dr, "f_{1*} = f_{1*}'", "\cong"'] & & H_n(D^n, \partial D^n)\, \mathbb{Z}_n \arrow[ll, "\cong"'] \arrow[d, "f_{1*}'"] \\
\mathrm{ldeg}(f_1)\, \mathbb{Z}_n H_n(D^n, \partial D^n) \arrow[r, "\cong"] & H_n(D^n, D^n \setminus f(x)) & H_n(D^n, D^n \setminus \mathring{Z}_1) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(D^n, \partial D^n)\, \deg(f_1')
\end{tikzcd}$$

If $\mathrm{ldeg}\, f_1 = 1 \Rightarrow \deg f_1' = 1 \overset{L.0}{\Rightarrow} f_1' \simeq id$ rel $\partial D^n \Rightarrow$

$\Rightarrow [g'] = \gamma_{1*}[g \circ f_1'] \overset{L3}{=} \gamma_{1*}[g \circ f]$

Case $k>1$.

$\boxed{A_4}$

Lemma 4. Given $U_i \subseteq f_i(\mathring{D}^n)$ $\exists f_i', g'$ s.t. $\gamma_{i*}[g\circ f_i] = \gamma_{i*}'[g'\circ f_i']$ & $f_i'(D^n) \subseteq U_i$

Pf: $0 \in Q_i \subseteq f_i^{-1}(U_i)$ concentric

Prop. $\Rightarrow$ $\exists g' \sim g$ (as pairs), $g\left(D^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{Q}_i)\right) \subseteq A$, $\gamma_*[g\circ f_i] = \gamma_*[g'\circ f_i]$

Let $H_i : D^n \times I \longrightarrow D^n$, $H_i(-,0) = id$, $H_i(-,1): D^n \xrightarrow{\cong} Q_i \subseteq D^n$,

$H_i(\partial D^n \times I) \subseteq D^n \setminus \mathring{Q}_i$. (Exist)

$f_i' := f_i \circ H(-,1)$ $\Rightarrow$ $f_i H(\partial D^n \times I) \subseteq g'^{-1}(A)$. Lemma 3 gives the result. $\boxed{4}$

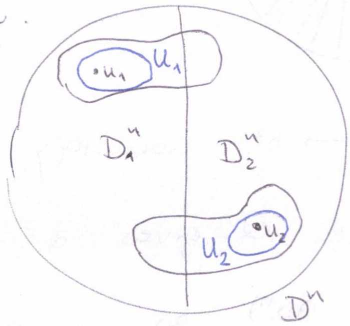
Induction on k. Let $u_i \in f_i(D^n)$ be chosen.

$D^n = D_1^n \cup D_2^n$, $\mathring{D}_1^n \cap \mathring{D}_2^n = \emptyset$, both

$\mathring{D}_1^n$ and $\mathring{D}_2^n$ contain some u_i

Let U_i be opens s.t. $u_i \in U_i \subseteq f_i(\mathring{D}^n)$

and $U_i \subseteq \mathring{D}_1^n$ or $U_i \subseteq \mathring{D}_2^n$



Lemma 4 $\Rightarrow$ wma $f_i(D^n) \subseteq D_1$ or $\subseteq D_2$ (otherwise replace)

Choose orientations for D_1^n, D_2^n s.t. $\mathrm{ldeg}(D_1^n \hookrightarrow D^n) = \mathrm{ldeg}(D_2^n \hookrightarrow D^n) = 1$

$$[g] = [g|_{D_1}] + [g|_{D_2}] \in \tilde{\pi}_n(X, A, x_0)$$

$$\parallel \qquad\qquad \parallel$$

$$\sum_{u_i \in D_1^n} \mathrm{ldeg}(f_i)\gamma_{i*}[g\circ f_i] + \sum_{u_i \in D_2^n} \mathrm{ldeg}(f_i)\gamma_{i*}[g\circ f_i]$$

$\boxed{4}$

This finishes the proof of A $\boxed{A}$

(B) 1

$C^{(m)}_k(X,A) = \{\sigma\in C_k(X,A) \mid \sigma(sk_m\Delta^k)\subseteq A\}$, $C^{(m)}_*(X,A)\subseteq C_*(X,A)$ subcomplex

<u>Prop.</u> X, A path-conn, $\pi_k(X,A,x_0)=0$ $\forall k<n$

$\Rightarrow$ $C^{(n-1)}_*(X,A) \hookrightarrow C_*(X,A)$ chain htp equivalence.

PROOF: [Bredon, VII. 10.5]

Define $p(\sigma): I\times\Delta^l \longrightarrow X$ for every singular simplex $\sigma:\Delta^l\longrightarrow X$

such that p satisfies:

(1) $p(\sigma)(0,-)=\sigma$

(2) $p(\sigma)(1,-)\in C^{(n-1)}_l(X,A)$ $\Big\}$ p is a homotopy connecting σ to an elt in $C^{(n-1)}$

(3) if $\sigma\in C^{(n-1)}_l(X,A)$ then $p(\sigma)(t,-)=\sigma$ $\forall t\in I$

(4) $p(\sigma)\circ(\underbrace{\iota_l\times d_i}_{\text{homology cross product}}) = p(\sigma\circ d_i)$ $\forall i=0,\dots,l$ where $\iota_l:\Delta^l\xrightarrow{id}\Delta^l$

Now we construct p. For $\sigma\in C^{(n-1)}_l(X,A)$, (3) already defines $p(\sigma)$

so assume $\sigma\in C_l(X,A)\setminus C^{(n-1)}_l(X,A)$. Induction on l.

<u>$l=0$:</u> $\Delta^0=*$, $\sigma(\Delta^0)\notin A$, let $p(\sigma)$ be a path connecting $\sigma(\Delta^0)$ to

some point in A.

$$\begin{tikzcd} \Delta^0 \arrow[r, "\sigma"] \arrow[d, "\cong"'] & X \\ 0\in I \arrow[ur, dashed, "p(\sigma)"'] & \end{tikzcd}$$

<u>$0<l\leq n-1$:</u> By induction we already have

$(I\times\partial\Delta^l, 1\times\partial\Delta^l) \xrightarrow{p(\partial\sigma)} (X,A)$ since $\partial\Delta^l=\bigcup_{i=0}^{l} d_i\Delta^{l-1}$

By (1) this agrees with $0\times\Delta^l \xrightarrow{\sigma} X$ on $0\times\partial\Delta^l$

so we have

$$\begin{tikzcd} (I\times\partial\Delta^l \cup 0\times\Delta^l, 1\times\partial\Delta^l) \arrow[r, "p(\partial\sigma)\cup\sigma"] \arrow[d, hook] & (X,A). \\ (I\times\Delta^l, 1\times\Delta^l) \arrow[ur, dashed, "\exists p(\sigma)"'] & \end{tikzcd}$$

$\pi_l(X,A,x_0)=0$ $\Rightarrow$

(We use $sk_{n-1}\Delta^l=\Delta^l$)

<u>$n\leq l$:</u> Same diagram, use HEP. Using $sk_{n-1}\Delta^l = sk_{n-1}\partial\Delta^l$

Easily checked: (1–4) are satisfied.

Let $\varphi: C_l(X,A) \longrightarrow C^{(n-1)}_l(X,A)$

$\sigma \longmapsto p(\sigma)(1,-)$

(4) $\Rightarrow$ φ is a chain map

Trivial by def: $C_*^{(n-1)}(X,A) \hookrightarrow C_*(X,A) \xrightarrow{\varphi} C_*^{(n-1)}(X,A)$ is the identity. (B) 2

Remains to see $\mathrm{incl} \circ \varphi \simeq \mathrm{id}$.

$$D: C_\ell(X,A) \longrightarrow C_{\ell+1}(X,A) \quad \text{chain homotopy}$$
$$\sigma \longmapsto p(\sigma)_*(z_1 \times z_\ell)$$

$$\begin{array}{ccc} & z_1\times z_\ell & & D(\sigma) & \\ C_{\ell+1}(I\times\Delta^\ell) & & \longrightarrow & & C_\ell(X) \\ \uparrow & & & & \\ C_\ell(I)\times C_\ell(\Delta^\ell) & & & & \\ (z_1, z_\ell) & & & & \end{array}$$

$$\begin{aligned} dD(\sigma) &= d\, p(\sigma)_*(z_1\times z_\ell) \\ &\overset{(4)}{=} p(\sigma)_*\left(d(z_1\times z_\ell)\right) \\ &= p(\sigma)_*(dz_1\times z_\ell - z_1\times dz_\ell) \quad \text{alg. cross prod. Leibniz} \\ &= p(\sigma)_*\left(0\times z_\ell - 1\times z_\ell - \sum_{i=0}^{\ell}(-1)^i z_1\times d_i\right) \end{aligned}$$

$$\begin{aligned} D(d\sigma) &= \sum_{i=0}^{\ell}(-1)^i D(\sigma\circ d_i) \\ &= \sum_{i=0}^{\ell}(-1)^i p(\sigma\circ d_i)_*(z_1\times z_{\ell-1}) \\ &= \sum_{i=0}^{\ell}(-1)^i p(\sigma)_*(z_1\times d_i) \end{aligned}$$

$$\Rightarrow dD(\sigma) + D(d\sigma) = p(\sigma)_*(0\times z_\ell - 1\times z_\ell) = \mathrm{id} - \varphi$$

$C_*^{(n-1)}(X,A)$ and $C_*(X,A)$ chain htp. equivalent

$\Rightarrow H_k(C_*^{(n-1)}(X,A)) \cong H_k(C_*(X,A)) = H_k(X,A) \quad \forall k$

$\forall k\le n-1$: $\forall \sigma\in C_k^{(n-1)}(X,A)$: $\sigma(sk_{n-1}\Delta^k)\subseteq A$, but $sk_{n-1}\Delta^k = \Delta^k \Rightarrow C_k^{(n-1)}(X,A)=0$

$\Rightarrow \underline{H_k(X,A) = 0 \;\; \forall k\le n-1}$, as desired.

To show the assertion about H_n, notice that $H_n(C_*^{(n-1)}(X,A)) \cong H_n(X,A)$ since $sk_{n-1}\Delta^n = \partial\Delta^n$

To see that h is an iso, we define an inverse) map

$$\psi_n : H_n\left(C_*^{(n-1)}(X,A)\right) \longrightarrow \tilde{\pi}(X,A,x_0)$$

On chains: $C_*^{(n-1)}(X,A) \longrightarrow \tilde{\pi}(X,A,x_0)$

$$\left(f: \Delta^n, \partial\Delta^n \to (X,A)\right) \longmapsto \gamma_*[f] \quad \text{where } \gamma \text{ is a path from } f(*) \text{ to } x_0$$
(using $\Delta^n \cong D^n$)

B3 Now we show that this descends to homology.

Since $C^{(n-1)}_{n-1}(X,A)=0$, $C^{(n-1)}_{n}(X,A)$ consists only of cycles.

We only need to show that ψ_n vanishes on boundaries.

Let $g : \Delta^{n+1} \longrightarrow X$, $g(\partial k_{n-1}\Delta^{n+1}) \subseteq A$, i.e. $g \in C^{(n-1)}_{n+1}(X,A)$.

$$\psi_n(\partial g) = \psi_n\Big(\sum_{i=0}^{n-1}(-1)^i g\circ d_i\Big) = \sum_{i=0}^{n-1}(-1)^i \psi_n(g\circ d_i) = \sum_{i=0}^{n-1}(-1)^i \gamma_{i*}[g\circ d_i]$$

Let $B := d_0\Delta^n \cup \dots \cup d_{n+1}\Delta^n. = \partial\Delta^{n+1}$

$$\bar g : (B, \partial B) \longrightarrow (B, \partial k_{n-1}B) \twoheadrightarrow (\partial\Delta^{n+1}, \partial k_{n-1}\Delta^{n+1}) \xrightarrow{g|_{\partial\Delta^{n+1}}} (X, A, x_0)$$

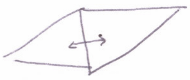
∂B in $\partial k_{n-1}\Delta^{n+1}$ is contractible

$\Rightarrow$ up to homotopy, $\bar g$ is

$$\begin{tikzcd} (B,\partial B) \arrow[r] & (\partial\Delta^{n+1}, *) \arrow[r, "g|_{\partial\Delta^{n+1}}"] \arrow[d] & (X,A) \\ & (*,*) \simeq (D^{n+1}, *) \arrow[ur] & \end{tikzcd}$$

$\Rightarrow$ $[\bar g]=0$ in $\tilde\pi(X,A,x_0)$.

$\mathrm{ldeg}(d_i) = -\mathrm{ldeg}(d_{i+1})$



$$\Rightarrow\ 0 = [\bar g] \underset{\substack{\text{HAT}\\(n)}}{=} \sum_{i=0}^{n-1} \mathrm{ldeg}(d_i)\, \gamma_{i*} \underbrace{[\bar g\circ d_i]}_{[g\circ d_i]} = \sum_{i=0}^{n-1}(-1)^i \gamma_{i*}[g\circ d_i] = \psi_n(\partial g)$$

$\Rightarrow$ ψ_n is well-def'd. Clearly ψ_n is inverse to h.

B